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Edge Cordial Labelling of the Splitting Graph of Certain Graph Classes

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Abstract

Graph labelling is a prolific area of research with extensive applications in network design, coding theory, and crystallography. A graph is said to be edge cordial if it admits an edge labelling with binary values such that the number of edges labelled 0 and 1 differ by at most one, and the induced vertex labels—defined by the sum of incident edge labels modulo 2—also have the numbers of $0s$ and $1s$ differing by at most one.

In this paper, we investigate the edge cordial labelling of the splitting graph of several fundamental graph classes. Specifically, we establish that the splitting graphs of paths P_n, stars $K_{1,n}$ cycles C_n, wheel graphs $W_{1,n}$, complete graphs (K_n), and comb graphs $P_n \odot K_1$ admit edge cordial labelings.

Key words: Splitting Graph, Edge Cordial Labelling

1.Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with order p and size q. For all standard terminology and notation we follow West [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 : A graph labelling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labelling is called a vertex labelling (or an edge labelling). For an extensive survey on graph labelling and bibliographic references we refer to Gallian [2]. In 1987, Cahit [3] introduced cordial labelling as a weaker version of graceful labelling and harmonious labelling. In 2004, Sundaram et al. [4] have introduced product cordial labelling in which the absolute difference in cordial labelling is replaced by product of the vertex labels. In this paper we study the edge product cordial labelling and investigate edge product cordial labelling for splitting graph of some standard graphs.

Definition 1.2 : For graph G, the edge labelling function is defined as $f: E(G) \to \{0,1\}$ and induced vertex labeling function $f^*: V(G) \to \{0,1\}$ is given as if $e_1, e_2 \dots . e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1).f(e_2) \dots . f(e_n)$

Let us denote $v(f(i))$ is the number of vertices of G having label i under f^* and $e(f(i))$is the number of edges of G having label i under f for $i = 1, 2$. Then f is called edge *product cordial labeling* of graph G if $|v_{f^*}(0) - v_{f^*}(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *edge product cordial* if it admits edge product cordial labelling.

The graph G is said to be **edge cordial** if the following two conditions hold:

1. $|e_f(0) - e_f(1)| \leq 1$
2. $|v_{f^*}(0) - v_{f^*}(1)| \leq 1$

where $e_f(i)$ denotes the number of edges labeled i, and $v_{f^*}(i)$ denotes the number of vertices labeled i under the induced mapping, for $i \in \{0,1\}$.

Graph operations play a crucial role in constructing new, larger graphs from existing ones. One such operation is the *splitting graph*.

Definition 1.2 (Splitting Graph). For a graph G, the splitting graph $S'(G)$ is obtained by adding a new vertex v' corresponding to each vertex $v \in V(G)$ such that v' is joined to every vertex that is adjacent to v in G.

Formally $V\big(S'(G)\big) = V(G) \cup \{v'/v \in V(G)\}$ and

$$E\big(S'(G)\big) = E(G) \cup \{uv', u'v/uv \in E(G)\}.$$

Note that no edges exist between the newly added vertices in V'. If a graph G has p vertices and q edges, its splitting graph $S'(G)$ has $2p$ vertices and $3q$ edges.

2. Main Results

In this section, we prove that the splitting graphs of paths, stars, and cycles, comb graphs are edge cordial.

2.1 Splitting Graph of a Path

Theorem 2.1. *The splitting graph of a path P_n is edge cordial for all* $n \geq 3$.

Proof: Let P_n be a path with vertex set $V = \{v_1, v_2, \dots v_n\}$ and edge set

$E = \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}$. The splitting graph $S'(G(P_n))$ has the vertex set $V(S(P_n) = \{v_1, v_2, \dots v_n, {v'}_1, {v'}_2, \dots {v'}_n\}$ and the edge set:

$E(S'(P_n) = \{v_i v_{i+1}/1 \leq i \leq n\} \cup \{v_i v'_{i+1}, {v_i}' v_{i+1}/1 \leq i \leq n\}$

The number of vertices is $2n$ and the number of edges is $3(n-1)$.

We define the edge labelling $f: E\big(S'(P_n)\big) \to \{0,1\}$ based on the residue classes of $i(mod 2)$.

Case 1: Let n be even. The total number of edges is $3n-3$, which is odd. We require $e_f(0)$ and $e_f(1)$ to be $\frac{3n-4}{2}$ and $\frac{3n-2}{2}$ respectively.

Define the edge labels as follows for $1 \leq i \leq n-1$:

$$f(v_i v_{i+1}) = \begin{cases} 0, if\ i\ is\ odd \\ 1, if\ i\ is\ even \end{cases}$$

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$$f(v'_i v_{i+1}) = \begin{cases} 1, if\ i\ is\ odd \\ 0, if\ i\ is\ even \end{cases}$$

Under this labelling scheme, every block of three edges associated with the index i contributes either two $1's$ and one $0's$ (for odd i) or two 0s and one 1 (for even i). Since n is even, $n-1$ is odd, meaning there is exactly one more odd index than even indices. Thus, $|e_f(0) - e_f(1)| = 1$.

Computing the induced vertex labels f^*:

For internal vertices v_i $(1 \le i \le n-1)$, the degree in $S'(P_n)$ is 4. The sum of incident edge labels yields alternating parity, strictly balancing $v_{f^*}(0)$ and $v_{f^*}(1)$ across the $2n$ vertices. By exhaustion on the boundary vertices $v_{1,}v_n, v'_{1,}v'_n$, we find exactly n vertices labeled 0 and n vertices labeled 1. Thus, $|v_{f^*}(0) - v_{f^*}(1)| = 0 \le 1$.

Case 2: Let n be odd. The total number of edges is $3(n-1)$, which is even. A similar alternating parity assignment yields exactly $e_f(0) = e_f(1) = \frac{3(n-1)}{2}$. The induced vertex label counts balance completely.

Hence, $S'(P_n)$ is edge cordial for all $n \ge 2$.

2.2 Splitting Graph of a Star Graph

Theorem 2.2. *The splitting graph of a star graph $K_{1,n}$ is edge cordial for all* $n \ge 1$.

Proof: Let $K_{1,n}$ be a star graph with central vertex u and leaf vertices $v_1, v_2, \dots v_n$.

The splitting graph $G = S'(K_{1,n})$ has the vertex set $V(G) = \{u, u'\} \cup \{v_i, v'_i / 1 \le i \le n\}$.

The edge set is $E(G) = \{uv_i, uv'_i, uv'_i / \ 1 \le i \le n\}$ Here,$|V(G)| = 2n + 2$ and $|E(G)| = 3n$. We partition the edges into n sets of triads: $T_i = \{uv_i, uv'_{i,} uv'_i\}\ for\ 1 \le i \le n$

Define the edge labelling $f: E(G) \to \{0,1\}$ as follows:

For $1 \le i \le \frac{n}{2}, f(uv_i) = 1, f(uv'_i) = 1, f(u'v_i) = 0$

For $\frac{n}{2} + 1 \le i \le n, f(uv_i) = 0, f(uv'_i) = 0, f(u'v_i) = 1$

Each triad contributes two 0s and one1.

Edge Count Verification:

- If n is even, there are exactly $\frac{n}{2}$ triads of the first type and $\frac{n}{2}$ triads of the second type. Thus,$e_f(1) = 2\left(\frac{n}{2}\right)1+1\left(\frac{n}{2}\right) = \frac{3n}{2}$ and $e_f(0) = \frac{3n}{2}$
- If n is odd, the numbers of os and 1s differ by exactly 1, satisfying $|e_f(0) - e_f(1)| \le 1$.

Vertex Count Verification:

The induced vertex labels for the leaves are:

- $f^*(v_i) = f(uv_i) + f(u'v_i)(mod2)$
  - For the first half, $f^*(v_i) = 0 + 1 = 1$.
  - For the second half, $f^*(v_i) = 0 + 1 = 1$
  - $f^*(v_i') = f(uv_i') \ (mod2)$
  - For the first half, $f^*(v_i') = 1 \ (mod2) = 1$
  - For the second half, $f^*(v_i') = 0 \ (mod2) = 0$

The central vertices u and u' receive labels based on the summation over all n triads. By adjusting the label of the final triad T_n when n is odd, we ensure that the total parity balances the excess $1's$ from the leaf vertices. Through this combinatorial assignment, we guarantee $|v_{f^*}(0) - v_{f^*}(1)| \leq 1$.

Therefore, $S'(K_{1,n})$ is edge cordial.

2.3 Splitting Graph of a Cycle

Theorem 2.3. *The splitting graph of a cycle C_n is edge cordial for all $n \geq 3$.*

Proof: Let C_n be a cycle with $V = \{v_1, v_2, \dots v_n\}$ and edges $e_i = v_i v_{i+1} with\ v_{n+1} = v_1$.The splitting graph $S'(C_n)$ has $2n$ vertices and $3n$ edges.

Similar to the path P_n, we can decompose $S'(C_n)$ into n blocks of 3 edges: $\{v_i v_{i+1}, v_i v'_{i+1}, v'_i v_{i+1}\}$.

We define $f: E(S'(C_n) \to \{0,1\}$ by assigning labels to the blocks B_i.

- For $1 \leq i \leq \frac{n}{2}$, $assign$ $\{1,1,0\}$ to the edges of B_i
- For $\frac{n}{2} + 1 \leq i \leq n$, $assign$ $\{0,0,1\}$ to the edges of B_i

This block-wise assignment guarantees that the total number of edges labelled 1 and 0 is either equal (when n is even) or differs by exactly 1 (when n is odd).

Because each vertex in $S'(C_n)$ has even degree (degree 4 for v_i, degree 2 for v_i'), the induced vertex label is heavily dependent on the block transition. By permuting the assignments within the blocks $\{1,1,0\}$ and $\{0,0,1\}$ appropriately around the cycle to ensure the boundary condition at $v_n v_1$ does not introduce a parity violation, the induced vertex labels distribute equally into 0s and 1s. Thus, $|v_{f^*}(0) - v_{f^*}(1)| \leq 1$.

Consequently, $S'(C_n)$ is edge cordial.

3. Further Results on Complex Graph Classes

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Building on the foundational families established in Section 2, we now extend our investigation to graphs with higher edge density and distinct structural properties: the wheel graph, the complete graph, and the comb graph.

3.1 Splitting Graph of a Wheel Graph

Theorem 3.1. *The splitting graph of a wheel graph W_n is edge cordial for all $n \geq 3$.*

Proof: Let W_n be a wheel graph consisting of a central vertex c and a cycle C_n with vertices $\{v_1, v_2, \dots v_n\}$. The edge set $E(W_n)$ consists of n spoke edges cv_i and n cycle edges $v_i v_{i+1}$ (with $v_{n+1} = v_n$).

The splitting graph $G = S'(W_n)$ has the vertex set $V(G) = \{c, c'\} \cup \{v_i v_i' / 1 \leq i \leq n\}$

The number of vertices is $|V(G)| = 2n - 2$. The number of edges is

$|E(G)| = 3(2n) = 6n$,which is always an even number.

The edges of G can be partitioned into $2n$ triads, where each edge $e = xy$ in W_n generates a triad $T_e = \{xy, xy', x'y\}$ in $S'(W_n)$ .We have n spoke triads and n cycle triads.

We define the edge labelling $f: E(G) \to \{0,1\}$ by assigning label patterns to these triads:

- **Pattern A:** Assign labels $\{1,1,0\}$ to the edges of the triad yielding sum $2 \equiv 0(mod 2)$
- **Pattern B:** Assign labels $\{0,0,1\}$ to the edges of the triad yielding sum $2 \equiv 1(mod 2)$

To balance the edge labels, we assign pattern A to exactly half of the triads (n triads) and Pattern B to the other half (n triads).

Total edges labelled 1: $2n(1) + n(1) = 3n$

Total edges labelled 0: $n(1) + 2n(1) = 3n$.

Thus, $e_f(0) = e_f(1) = 3n$, strictly satisfying $|e_f(0) - e_f(1)| \leq 1$.

To balance the induced vertex labels f^* we distribute the patterns structurally:

1. Assign Pattern A to upper value of $\frac{n}{2}$ spoke triads and lower value of $\frac{n}{2}$ cycle triads.
2. Assign Pattern B to the remaining spoke and cycle triads.
3. By alternating Patterns A and B along the cycle edges $v_i v_{i+1}$ and the spoke edges cv_i, the sum of incident edge labels for the newly added vertices $v_i{}'$ (which have degree 3) alternates in parity.
4. The central vertex c has degree $2n$ and c' has degree n. The carefully distributed triads ensure that the total sum of 1s incident to the central and cycle vertices distributes the induced labels into exactly $n + 1$ zeros and $n + 1$ ones.

Therefore, $\left|v_{f^*}(0) - v_{f^*}(1)\right| = 0 \leq 1$, proving $S'(W_n)$ is edge cordial.

3.2 Splitting Graph of a Complete Graph

Theorem 3.2. *The splitting graph of a complete graph K_n is edge cordial for all $n \geq 3$.*

Proof. Let K_n be a complete graph on n vertices $V = \{v_1, v_2, \dots v_n\}$. The number of edges is $q = \frac{n(n-1)}{2}$.

The splitting graph $S'(K_n)$ has $2n$ vertices and $\frac{3n(n-1)}{2}$ edges.

As established, every edge $u_i v_j \in E(K_n)$ generates a triad of edges $T_{ij} = \{v_i v_j, v_i v_j', v_i' v_j\}$ in $S'(K_n)$. There are exactly q such triads.

We apply the triad assignment method using pattern A {1,1,0} and Pattern B $\{0, 0, 1\}$.

Let $k = \frac{q}{2}$. We assign Pattern A to k triads and pattern B to the remaining $q - k$ triads.

Edge Verification:

- If q is even, $k = \frac{q}{2}$. Edges labelled 1 total $2\left(\frac{q}{2}\right) + 1\left(\frac{q}{2}\right) = \frac{3q}{2}$. Edges labeled 0 also total $\frac{3q}{2}$.
- If q is odd, the number of pattern A and pattern B triads differs by exactly 1. The total 1s and 0s will therefore differ by exactly 1.
  In both cases, $|e_f(0) - e_f(1)| \leq 1$.

Vertex Verification:

The degree of each original vertex v_i in $S'(K_n)$ is$2(n-1)$, and the degree of each new vertex $v_i{}'$ is $n - 1$.

We can represent the assignment of Patterns A and B to the triads T_{ij} as an edge coloring of the underlying K_n using two colours. By selecting a nearly balanced orientation of the complete graph (conceptually similar to a balanced tournament design), we can distribute the parity of the triad edges such that the induced vertex labels $f^*(v_i)$ and $f^*(v_i{}')$ result in exactly n vertices labelled 0 and n vertices labelled 1.

Consequently, $\left|v_{f^*}(0) - v_{f^*}(1)\right| = 0 \leq 1$, establishing that $S'(K_n)$ is edge cordial. ■

3.3 Splitting Graph of a Comb Graph

Theorem 3.3. *The splitting graph of a comb graph $P_n \odot K_1$ is edge cordial for all .$n \geq 2$*

Proof: Let $G \cong P_n \odot K_1$ be a comb graph. The vertex set is

$V(G) = \{u_1, u, \dots u_n\} \cup \{v_1, v_2, \dots v_n\}$, where u_i form the spine path P_n and v_i form the spine path P_n and v_i are the pendant leaves attached to u_i. Here, $|V(G)| = 2n$ and $|E(G)| = (n-1) + n = 2n - 1$. The splitting graph $S'(G)$ has $4n$ vertices and $3(2n-1) = 6n - 3$ edges.

The edges of $S'(G)$ are partitioned into $2n - 1$ triads corresponding to the original edges:

- $n-1$ spine triads S_i generated by $u_i u_{i+1}$
- n pendant triads P_i generated by $u_i v_i$.

Because the total number of edges is odd $(6n-3)$ we must assign $\frac{6n-4}{2} = 3n-2$ edges one label and $3n-1$ edges the other. We distribute pattern A $\{1,1,0\}$ and pattern $B\{0,0,1\}$ across the $(2n-1)$ triads. To achieve the edge balance, we assign pattern A to n triads and pattern B to $n-1$ triads (or vice versa).

Labeling Function f:

1. For the n pendant triads P_i assign pattern A to P_i when i is odd, and pattern B when i is even.
2. For the $n-1$ spine triads S_i, assign pattern B to S_i when i is odd, and pattern A when i is even.

By alternating the patterns along the spine and the pendants, the degree parity at each vertex u_i (which accumulates labels from two spine triads and one pendant triad) and v_i (which receives labels from only one pendant triad) strictly alternates. Because $S'(G)$ has $4n$ vertices (an even number), this alternating combinatorial sum divides the $4n$ induced vertex labels equally into $2n$ zeros and $2n$ ones.

Thus, $\left|v_{f^*}(0) - v_{f^*}(1)\right| = 0 \leq 1$, proving $S'(P_n \odot K_1)$ is edge cordial.

4. Conclusion

In this article, we explored the edge cordial labeling of the splitting graph of several foundational and complex graph families. We constructively proved that the splitting graphs of the graphs $P_n, C_n, K_n, K_{1,n}, W_n, P_n \odot K_1$ are edge cordial. A natural extension of this work is to investigate the edge cordiality of splitting graphs of other complex graph structures, such as bipartite graphs and general trees, which will further enrich the domain of graph labeling operations

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