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Solution of Burger-Fisher Type Equation Via Variational Iteration Techniques with He's Polynomials

T. K. Kumkar

Department of Mathematics,
Arts, Science and Commerce College, Rahata, Dist-Ahmednagar (MS)-423107, India

Abstract

In this article, coupling of He's variational iteration techniques (HVIT) and Homotopy Perturbation Method (HPM) is used for solving nonlinear partial differential (NPD) Burger-Fisher type equation. In this method as similar to HVIT,we have to construct correction functional and Lagrange multiplier is calculated optimally by using variational theory. Lastly He's polynomials are introduced in the correction functional and equating likewise powers of p gives solution of NPD equation. The complete advantages of HVIT and HPM are taken by this technique. It is also known as Variational iteration techniques with He's Polynomials (VITHp). This technique is free from rounding off errors. Also it is applied without any restrictive assumption, discretization and transformation.

Keywords: variational iteration techniques, He's polynomials, Homotopy Perturbation Method, Burger-Fisher equation, nonlinear Partial differential equation.

1. Introduction:

The variational iteration technique [1,2,3] was first proposed by Ji-Huan He in 1999.The VIT does not involve discretization also it does not require large number of computer memory. Therefore, it is preferable to many numerical problems as it is free from rounding off errors. The VIT is reliable and efficient for wide variety of scientific and engineering applications. This method is applied to linear, nonlinear, ordinary and partial differential equations [5]. This method solves large class of nonlinear problems [4] easily, effectively and accurately with approximations converging rapidly to exact solutions.

In this paper, VITHp is implemented for solving some nonlinear partial differential Burger-Fisher (NPDBF) equation.The Paper has been organized as follows: In section 2, HPM is introduced. He's Polynomials is also discussed in this section. In section 3, HVIT with He's polynomials and its application for Burger-Fisher type equation is analyzed. The last section deals with conclusion.

2. Homotopy Perturbation Method (HPM):

Homotopy Perturbation Method [6] is proposed by Ji-huan He. HPM is nothing but combination of Standard Homotopy method and Perturbation method.

Consider the NPD equation, $F(v) - g(s) = 0,\ s \in Q$ (1)

Subject to the boundary condition, $G\left(v, \frac{\partial v}{\partial k}\right) = 0,\ s \in \Psi$

where F is differential operator, G is boundary operator, $g(s)$ is a function, Ψ is the boundary of domain Q.

Equation (1) can be rewritten by dividing operator F into linear (L) and nonlinear (N) parts as

$$L(v) + N(v) - g(s) = 0,\ s \in Q \quad (2)$$

Construct Homotopy $w(s,q): Q \times [0,1] \to R$ which satisfies homotopic structure,

$$H(w,q) = (1-q)[L(w) - L(v_0)] + q[L(w) + N(w) - g(s)] = 0, s \in Q \quad (3)$$

Here v_0 is initial approximation by which above boundary condition holds and $q \in [0,1]$.

If we substitute $q = 0$ and $q = 1$ in equation (3) we get,

$$H(w,0) = L(w) - L(v_0) = 0$$

$$H(w,1) = L(w) + N(w) - g(s) = 0 \quad (4)$$

Hence from equation (4), it is observe that $W(s,q)$ is just change from $v_0(s)$ to $v(s)$ due to changing process of q from 0 to 1.this is known as deformation in topology.

Now, equation (3) can be written as a power series of q by considering q as small parameter as

$$w = w_0 + qw_1 + q^2 w_2 + q^3 w_3 + \cdots, \quad (5)$$

Therefore we get approximate solution of equation (2) as follows,

$$v = \lim_{q \to 1} w = w_0 + w_1 + w_2 + \cdots, \quad (6)$$

2.1.He's Polynomials:

Consider the functional equation in the form, $v - N(v) - g = 0,$ (7)

Where function v is unknown function which is to be determine, N is nonlinear operator from Hilbert space H to H and g is source term or known function.

Consider equation (1) as, $T(w) = w(x) - g(x) - N(w) = 0,$

Define homotopy function $H(w,q)$ as: $H(w,0) = R(w),\ \ H(w,1) = T(w)$

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Here $R(w)$ is integral operator with solution w_0.

We can select convex Homotopy

$$H(w,q) = (1-q)R(w) + qT(w) = 0 \qquad (8)$$

Here the parameter $q \in [0,1]$.

We can write equation (5) as, $w = w_0 + qw_1 + q^2w_2 + q^3w_3 + \cdots, = \sum_{r=0}^{\infty} q^r w_r$ (9)

Hence solution of equation (6) is , $v = \lim_{q\to 1} w = w_0 + w_1 + w_2 + \cdots, = \sum_{r=0}^{\infty} w_r$

hence, $T(w) = R(w) - N(w)$ $\quad \because R(w) = w(x) - g(x)$

Equation (8) becomes,

$$H(w,q) = (1-q)R(w) + q[R(w) - N(w)] = 0$$

$$= w(x) - g(x) - qN(w) = 0 \qquad (10)$$

Maclaurin's expansion of $N(w)$ with respect to q is,

$$N(w) = N(w)|_{q=0} + q\left(\frac{\partial}{\partial q}N(w)\Big|_{q=0}\right) + \frac{q^2}{2!}\left(\frac{\partial^2}{\partial q^2}N(w)\Big|_{q=0}\right) + \frac{q^3}{3!}\left(\frac{\partial^3}{\partial q^3}N(w)\Big|_{q=0}\right) + \cdots$$

$$+ \frac{q^n}{n!}\left(\frac{\partial^n}{\partial q^n}N(w)\Big|_{q=0}\right) + \cdots$$

In above equation put value of w from equation (9), we get

$$N(w) = N\left(\sum_{r=0}^{\infty} q^r w_r\right)\Bigg|_{q=0} + q\left(\frac{\partial}{\partial q}N\left(\sum_{r=0}^{\infty} q^r w_r\right)\Bigg|_{q=0}\right) + \frac{q^2}{2!}\left(\frac{\partial^2}{\partial q^2}N\left(\sum_{r=0}^{\infty} q^r w_r\right)\Bigg|_{q=0}\right) + \cdots$$

$$N(w) = N(w_0) + q\left(\frac{\partial}{\partial q}N\left(\sum_{r=0}^{1} q^r w_r\right)\Bigg|_{q=0}\right) + \frac{q^2}{2!}\left(\frac{\partial^2}{\partial q^2}N\left(\sum_{r=0}^{2} q^r w_r\right)\Bigg|_{q=0}\right)$$

$$+ \frac{q^3}{3!}\left(\frac{\partial^3}{\partial q^3}N\left(\sum_{r=0}^{3} q^r w_r\right)\Bigg|_{q=0}\right) + \cdots \qquad (11)$$

Substitute values of equations (9) and (11) in equation (10) we get,

$$H(w,q) = \sum_{r=0}^{\infty} q^r w_r - g(x) - q\left[N(w_0) + q\left(\frac{\partial}{\partial q} N\left(\sum_{r=0}^{1} q^r w_r\right)\Bigg|_{q=0}\right) + \frac{q^2}{2!}\left(\frac{\partial^2}{\partial q^2} N\left(\sum_{r=0}^{2} q^r w_r\right)\Bigg|_{q=0}\right) + \cdots + \frac{q^n}{n!}\left(\frac{\partial^n}{\partial q^n} N\left(\sum_{r=0}^{n} q^r w_r\right)\Bigg|_{q=0}\right) + \cdots\right]$$

From the above equation equating the terms of identical powers of q,we get

$$q^0: w_0(x) - g(x) = 0 \qquad w_0(x) = g(x)$$

$$q^1: w_1(x) - N(w_0) = 0 \qquad w_1(x) = N(w_0)$$

$$q^2: w_2(x) - \frac{\partial}{\partial q} N\left(\sum_{r=0}^{1} q^r w_r\right)\Bigg|_{q=0} = 0 \qquad w_2(x) = \frac{\partial}{\partial q} N\left(\sum_{r=0}^{1} q^r w_r\right)\Bigg|_{q=0}$$

$$q^3: w_3(x) - \frac{1}{2!}\frac{\partial^2}{\partial q^2} N\left(\sum_{r=0}^{2} q^r w_r\right)\Bigg|_{q=0} = 0 \qquad w_3(x) = \frac{1}{2!}\frac{\partial^2}{\partial q^2} N\left(\sum_{r=0}^{2} q^r w_r\right)\Bigg|_{q=0}$$

$\vdots$

and so on.

Now we define the He's Polynomials as:

$$H_n(w_0, w_1, \dots, w_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} N\left(\sum_{r=0}^{n} q^r w_r\right)\Bigg|_{q=0}, \qquad n = 0,1,2,\dots$$

Hence He's Polynomials are,

$$H_0(w_0) = N(w_0)$$

$$H_1(w_0, w_1) = \frac{\partial}{\partial q} N\left(\sum_{r=0}^{1} q^r w_r\right)\Bigg|_{q=0} = \frac{\partial}{\partial q} N(w_0 + q w_1)\Big|_{q=0}$$

$$H_2(w_0, w_1, w_2) = \frac{1}{2!}\frac{\partial^2}{\partial q^2} N\left(\sum_{r=0}^{2} q^r w_r\right)\Bigg|_{q=0} = \frac{1}{2!}\frac{\partial^2}{\partial q^2} N(w_0 + q w_1 + q^2 w_2)\Big|_{q=0}$$

$\vdots$

The nonlinear function $N(v)$ can be express in terms of He's polynomials as:

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$$N(v)=\sum_{n=0}^{\infty}H_n(w_{0,}w_1,\dots,w_n)$$

$$=H_0(w_0)+H_1(w_{0,}w_1)+H_2(w_{0,}w_1,w_2)+\cdots+H_n(w_{0,}w_1,\dots,w_n)+\cdots$$

Therefore from equation (7) the solution is express as,

$$v(x)=g(x)+N(v)=g(x)+\sum_{n=0}^{\infty}H_n(w_{0,}w_1,\dots,w_n)$$

Illustration:

Consider NPD equation $v'+vv'=1+x, v(0)=0$ (12)

Where (′) represent derivative of function v.Also the term vv' is nonlinear.

Construct homotopy structure of equation (12) as:

$$H(w,q)=w'(x)-(1+x)+qw(x)w'(x)=0$$

$$H(w,q)=(w_0'+qw_1'+q^2w_2'+q^3w_3'+\cdots)-(1+x)+q.(w_0+qw_1+q^2w_2+q^3w_3+\cdots).(w_0'+qw_1'+q^2w_2'+q^3w_3'+\cdots)=0$$

Collecting powers of q ,we get

$$(w_0'-1-x)q^0+(w_1'+w_0w_0')q^1+(w_2'+w_0w_1'+w_1w_0')q^2+(w_3'+w_0w_2'+w_1w_1'+w_2w_0')q^3+(w_4'+w_0w_3'+w_1w_2'+w_2w_1'+w_3w_0')q^4+\cdots$$

Hence here He's polynomials are as follows:

$$H_0(w_0)=w_0w_0'$$

$$H_1(w_{0,}w_1)=w_0w_1'+w_1w_0'$$

$$H_2(w_{0,}w_1,w_2)=w_0w_2'+w_1w_1'+w_2w_0'$$

$$H_3(w_{0,}w_1,w_2,w_3)=w_0w_3'+w_1w_2'+w_2w_1'+w_3w_0'$$

$\vdots$

3. Variational Iteration Techniques With He's Polynomials:

The Variational iteration techniques with He's Polynomials(VITHp) [7] is coupling of HVIT with HPM. Actually it is coupling of correction functional in HVIT with He's Polynomials. We will discuss this technique as follows.

Consider the general NPD equation,

$$Lv+Nv=g(x) \quad (13)$$

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here L and N are linear and nonlinear operator respectively.Also $g(x)$ is source term.

We construct correction functional of equation (13) as,

$$v_{n+1} = v_n + \int_0^t \lambda[Lv_n + N\overline{v_n} - g(s)]ds \qquad (14)$$

Here $\overline{v_n}$ is restricted variation therefore $\delta\overline{v_n} = 0$.As previous chapter, we can determine Lagrange multiplier $\lambda(s,t)$ optimally via variational theory. Moreover value of $\lambda(s,t)$ is also obtainby using following formula, $\lambda = \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}$ where n is degree of given NPD equation (13)

Now apply HPM to correction functional which is obtain after substituting Lagrange multiplier $\lambda(s)$ in equation (14) as

$$\sum_{n=0}^{\infty} q^n\, v_n = v_0 + q\int_0^t \lambda\left[\sum_{n=0}^{\infty} q^n\, Lv_n + \sum_{n=0}^{\infty} q^n\, N\overline{v_n}\right]ds - \int_0^x \lambda(s)g(s)ds \qquad (15)$$

The above equation (15) is coupling of correction functional in HVIT and He's Polynomials.By using this equation (15) we compare like powers of q which gives solution and by using these solutions we get series form of solution of equation (13)

3.1. Solution of Burger-Fisher Equation By VITHp:

In this section we will discuss nonlinear Partial differential equation which is Burger-Fisher type (NPDBF) [8] equation.

The family of nonlinear equation in the form, $v_t - v_{xx} = f(v)$ (16)

If function $f(v) = -v.v_x$ in equation (16), we get Burgers equation as,

$$v_t - v_{xx} + v.v_x = 0$$

If function $f(v) = v(1-v)$ in equation (16), we get Fisher equation as,

$$v_t - v_{xx} = v(1-v)$$

We obtain Burger-Fisher equation by substituting $f(v) = v.v_x + v(1-v)$ in equation (16) as,

$$v_t - v_{xx} = v.v_x + v(1-v)$$

Hence NPDBF equation is,

$$v_t - v_{xx} - vv_x - v(1-v) = 0 \qquad (17)$$

The exact solution of the above NPDBF equation is given as

$$v(x,t) = \frac{1}{2}\left\{1 + tanh\left[\frac{1}{4}\left(x + \frac{5}{2}t\right)\right]\right\} \qquad (18)$$

We get initial approximation by taking $t = 0$ in above equation as,

$$v(x,0) = v_0 = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{4}\right)\right]$$

We can construct correction functional of equation (17) by using equation (14)

and $\lambda = -1$ as, $v_{n+1} = v_n - \int_0^t \left[\frac{\partial v_n}{\partial s} - \frac{\partial^2 \overline{v_n}}{\partial x^2} - \overline{v_n}\frac{\partial \overline{v_n}}{\partial x} - \overline{v_n} + [\overline{v_n}]^2\right] ds$ (19)

By applying Homotopy Perturbation method to equation (19) we get,

$$\sum_{n=0}^{\infty} q^n v_n = v_0 + q\int_0^t \left[\begin{array}{c}\sum_{n=0}^{\infty} q^n \frac{\partial^2 v_n}{\partial x^2} + \left(\sum_{n=0}^{\infty} q^n v_n\right)\left(\sum_{n=0}^{\infty} q^n \frac{\partial v_n}{\partial x}\right) \\ + \sum_{n=0}^{\infty} q^n v_n - \left(\sum_{n=0}^{\infty} q^n v_n\right)^2\end{array}\right] ds$$

Equating coefficients of powers of q ,with the help of Maxima software we get,

$$q^0: v_0(x,t) = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{4}\right)\right],$$

$$q^1: v_1 = \int_0^t \frac{\partial^2 v_0}{\partial x^2} ds + \int_0^t v_0 \frac{\partial v_0}{\partial x} ds + \int_0^t v_0 ds - \int_0^t {v_0}^2 ds$$

$$= -\frac{t\,\mathrm{sech}\left(\frac{x}{4}\right)^2 \tanh\left(\frac{x}{4}\right)}{16} + \frac{t\,\mathrm{sech}\left(\frac{x}{4}\right)^2 \left(\tanh\left(\frac{x}{4}\right) + 1\right)}{16} + \frac{t\left(\tanh\left(\frac{x}{4}\right) + 1\right)}{2} - \frac{t\left(\tanh\left(\frac{x}{4}\right) + 1\right)^2}{4}$$

$$= \frac{5t}{16\cosh\left(\frac{x}{4}\right)^2} = \frac{5t}{16}\,\mathrm{sech}^2\left(\frac{x}{4}\right)$$

$$q^2: v_2 = \int_0^t \frac{\partial^2 v_1}{\partial x^2} ds + \int_0^t v_0 \frac{\partial v_1}{\partial x} ds + \int_0^t v_1 \frac{\partial v_0}{\partial x} ds + \int_0^t v_1 ds - 2\int_0^t v_0 v_1 ds,$$

$$= -\frac{25t^2}{128}\,\mathrm{sech}^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)$$

$$q^3: v_3 = \int_0^t \frac{\partial^2 v_2}{\partial x^2} ds + \int_0^t v_0 \frac{\partial v_2}{\partial x} ds + \int_0^t v_1 \frac{\partial v_1}{\partial x} ds + \int_0^t v_2 \frac{\partial v_0}{\partial x} ds + \int_0^t v_2 ds - 2\int_0^t v_0 v_2 ds - \int_0^t {v_1}^2 ds$$

$$= \frac{125t^3}{1536} \operatorname{sech}^2\left(\frac{x}{4}\right)\left[1 - \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{4}\right)\right]$$

Continuing in this way we get other approximations.

Therefore,solution of NPDBF equation (17) is,

$$v = v_0 + v_1 + v_2 + \cdots$$

$$v = \frac{1}{2}\left[1 + \tanh\left(\frac{x}{4}\right)\right] + \frac{5t}{16}\operatorname{sech}^2\left(\frac{x}{4}\right) - \frac{25t^2}{128}\operatorname{sech}^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right) + \frac{125t^3}{1536}\operatorname{sech}^2\left(\frac{x}{4}\right)\left[1 - \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{4}\right)\right] + \cdots \quad (20)$$

Graphical representation of VITHp (V_M) solution (20)is given as,

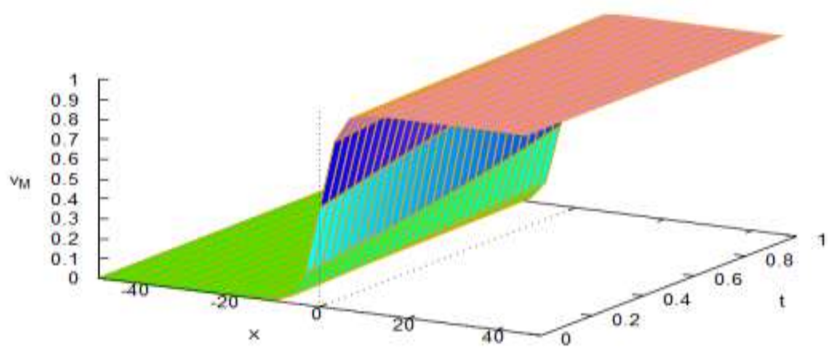


Figure 1: V_M solution of NPDBF equation (17)

Also,Exact solution V_e (18) of NPDBF equation is represented graphically as,

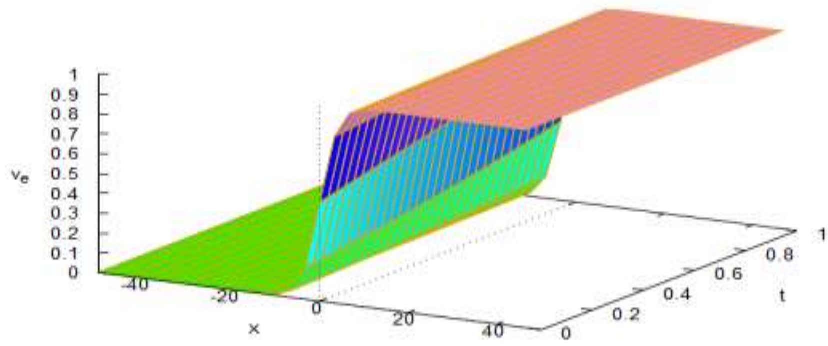
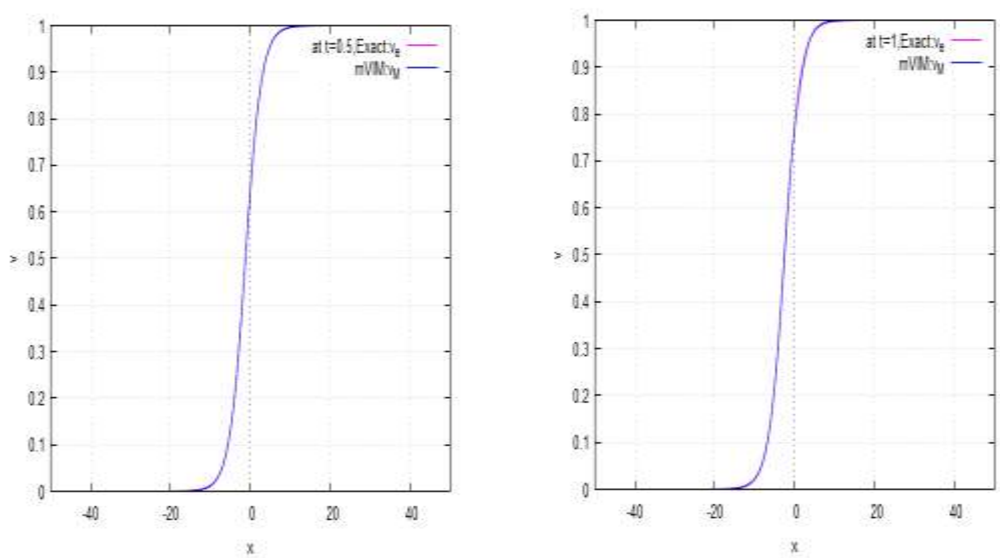


Figure 2: Exact solution V_e of NPDBF equation (17)



(a) (b)

Figure 3: V_M and V_e solution of NPDBF equation at (a) t =0.5 (b) t =1

Table : Absolute errors between VITHp and exact solution of NPDBFequation:

x	VITHp Solution (v_m)	Exact Solution (v_e)	Absolute error	VITHp Solution (v_m)	Exact Solution (v_e)	Absolute error
	t = 0.5			t = 1		
-50	$2.58455x10^{-11}$	$2.59461x10^{-11}$	$1.0057x10^{-13}$	$4.6618x10^{-11}$	$4.8473x10^{-11}$	1.855x10-^{12}
-40	$3.83581x10^{-9}$	$3.85074x10^{-9}$	$1.4930x10^{-11}$	$6.91882x10^{-9}$	$7.19413x10^{-9}$	$2.753x10^{-10}$

-30	$5.69285x10^{-7}$	$5.71500x10^{-7}$	$2.2150x10^{-9}$	$1.02684x10^{-6}$	$1.06770x10^{-6}$	$4.086x10^{-8}$
-20	$8.44825x10^{-5}$	$8.48110x10^{-5}$	$3.2850x10^{-7}$	0.0001523781	0.000158436	$6.0579x10^{-6}$
-10	$12.3886x10^{-3}$	$12.43165x10^{-3}$	$4.298x10^{-5}$	0.0222049528	0.022977370	$7.7242x10^{-4}$
0	0.6511637400	0.6513548600	$1.9113x10^{-4}$	0.7718098958	0.777299861	$5.4899x10^{-3}$
10	0.9964408400	0.9964063970	$3.4439x10^{-5}$	0.9985707075	0.998073265	$4.9744x10^{-4}$
20	0.9999759550	0.9999756997	$2.5585x10^{-7}$	0.9999906578	0.999986993	$3.6649x10^{-6}$
30	0.9999998379	0.9999998363	$1.724x10^{-9}$	0.9999999371	0.999999912	$2.4800x10^{-8}$
40	0.9999999989	0.9999999989	$1.1628x10^{-11}$	0.9999999996	0.999999999	$2x10^{-10}$
50	1.000000000	1.000000000	$7.8159x10^{-14}$	1.000000	1.000000	$3.999x10^{-12}$

Conclusions:

In this article, we have used VITHp for nonlinear partial differential (NPD) Burger-Fisher type equation. From the above table, absolute error between VITHp solution and exact solution of NPDBF equation very small.Hence the result obtained by VITHp has good agreement with exact solution. therefore solution obtained by VITHp is accurate. Hence this method gives better accuracy.

References:

1. Aasaraai, A. (2015). New explicit and Soliton wave solutions of some nonlinear partial differential equations with Infinite Series method. Journal of Sciences, Islamic Republic of Iran,26(4):349-354.

2. Abassy,T.(2010).Modified Variational Iteration Method (nonlinear homogeneous initial value problem).Computers and Mathematics with applications 59:912-918.

3. Abassy,T.(2012).New applications of the Modified Variational Iteration Method.Studies innonlinear science,3(2):49-61.

4. Abassy,T.,El-Tawil,M.et al.(2007). Modified Variational Iteration Method for Boussinesq equation. An International Journal of Computers and Mathematics with applications,54:955-965.

5. Abassy,T.,Tawil,M.,et al.(2007).Exact solution of some nonlinear partial differential equations using the Variational Iteration method linked with Laplace transform and Pade technique. An International Journal of Computers and Mathematics with applications 54:940-954.

6. Easif,F.,Manaa,S.,et al.(2015).Variational Homotopy Perturbation Method for solving Benjamin –Bona –Mahony equation.Applied Mathematics,:675-683.

7. Matinfar, M.and Ghasemi, M. (2013). Construction of solitary solution and compacton-like solution by the variational iteration method using He's polynomials.Journal of Computational and Applied Research in Mechanical Engineering,2(2):59-68.

8. Moghimi,M.and Hejazi,F.(2007).Variational Iteration Method for solving generalized Burgers-Fisher and Burgers equations.Chaos solitons and Fractals,33:1756-1761.